

ON THE TRACES OF ELEMENTS OF MODULAR GROUP

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ABSTRACT. We prove a conjecture by W. Bergweiler and A. Eremenko on the traces of elements of modular group in this paper.

1. INTRODUCTION

W. Bergweiler and A. Eremenko made a remarkable conjecture on the traces of elements of modular group in [1]. The main result of this paper is to prove their conjecture. We expect this result to have future applications in some fields such as control theory.

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. These two matrices generate the free group which is called $\Gamma(2)$, the principal congruence subgroup of level 2. With arbitrary integers $m_j \neq 0, n_j \neq 0$, consider the trace of the product

$$p_k(m_1, n_1, \dots, m_k, n_k) = \text{tr}(A^{m_1} B^{n_1} \dots A^{m_k} B^{n_k}).$$

It is easy to see that p_k is a polynomial in $2k$ variables with integer coefficients. This polynomial can be written explicitly though the formula is somewhat complicated.

Choosing an arbitrary sequence σ of $2k$ signs \pm , we make a substitution

$$p_k^\sigma(x_1, y_1, \dots, x_k, y_k) = p_k(\pm(1 + x_1), \pm(1 + y_1), \dots, \pm(1 + x_k), \pm(1 + y_k)).$$

Our main theorem is the following one.

Theorem 1.1. *The polynomial p_k , for every $k > 0$, has the property that for every σ , all the coefficients of the polynomial p_k^σ are of the same sign, that is, the sequence of coefficients of p_k^σ has no sign changes.*

which was conjectured by W. Bergweiler and A. Eremenko in [1].

We prove the theorem by induction on k . However it is not easy to pass from “level k ” to “level $k + 1$ ” since that p_k has the above property does not simply imply that p_{k+1} has the same one. The idea here is to substitute p_k ’s with a suitable set of polynomials containing the p_k ’s so that the difficulty disappears. This idea is explained in section 2 (see Proposition 2.2) and the theorem is showed in section 3.

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2. TRACES

2.1. **Good polynomials.** Set

$$F_k = \begin{pmatrix} f_k & h_k \\ t_k & g_k \end{pmatrix} = A^{x_1} B^{y_1} A^{x_2} B^{y_2} \dots A^{x_k} B^{y_k}$$

where $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. Then the trace $p_k = \text{tr} F_k = f_k + g_k$ and all f_k, h_k, t_k, g_k are the polynomials in $2k$ variables $x_1, y_1, \dots, x_k, y_k$ with integer coefficients whose explicit formula can be found in [1].

A sequence σ of $2k$ signs \pm can be viewed as a function $\sigma : \{1, 2, \dots, 2k\} \rightarrow \{1, -1\}$. For any polynomial f in variables $x_1, y_1, \dots, x_k, y_k$, set

$$f^\sigma = f(\sigma(1)(1 + x_1), \sigma(2)(1 + y_1), \dots, \sigma(2k-1)(1 + x_k), \sigma(2k)(1 + y_k))$$

Definition 2.1. A polynomial f in $2k$ variables is said to be good if for arbitrary sequence σ of $2k$ signs, all the coefficients of f^σ have the same sign.

Let $\text{Mat}(2, 2)$ be the set of 2×2 matrices over \mathbf{R} , the set of real numbers. Denote by F_k^σ the matrix $\begin{pmatrix} f_k^\sigma & h_k^\sigma \\ t_k^\sigma & g_k^\sigma \end{pmatrix}$. If $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \text{Mat}(2, 2)$, then

$$\begin{aligned} \text{tr}(F_k M) &= af_k + bh_k + ct_k + dg_k \\ \text{tr}(F_k^\sigma M) &= af_k^\sigma + bh_k^\sigma + ct_k^\sigma + dg_k^\sigma \end{aligned}$$

Write

$$\begin{aligned} A_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & A_2 &= \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} & A_3 &= \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}, \\ A_4 &= \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} & A_5 &= \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} & A_6 &= \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}. \end{aligned}$$

Note that

$$(2.1) \quad A_4 + A_5 = 4A_2, \quad A_4 + A_6 = 4A_3, \quad A_4^t + A_5 = 4A_2^t, \quad A_4^t + A_6 = 4A_3^t, \quad A_2 + A_3 = 4A_1,$$

$$(2.2) \quad A_4 = -A^{-1}B^{-1}, \quad A_4^t = -AB, \quad A_5 = AB^{-1}, \quad A_6 = A^{-1}B.$$

Let S be a subset of $\text{Mat}(2, 2)$, we have

Proposition 2.2. If S satisfies that

$$\mathbf{P1):} \quad a > 0, \text{ for all } M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in S,$$

$$\mathbf{P2):} \quad \text{tr}(CM) \geq 0, \text{ for each } C \in \{A_4, A_4^t, A_5, A_6\}, M \in S, \text{ where } D^t \text{ stands for the transpose of the matrix } D,$$

$$\mathbf{P3):} \quad CS \subseteq S, \text{ for each } C \in \{A_4, A_4^t, A_5, A_6\},$$

then $af_k + bh_k + ct_k + dg_k$ is good, for every $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in S, k \geq 1$.

Remark 2.3. S satisfies the conditions P1), P2) P3) if and only if so does the cone $\text{Cone}(S) \triangleq \{\sum a_i M_i \mid a_i \geq 0, M_i \in S\}$. Furthermore, any set S satisfying P1) possesses the property that $af_k + bh_k + ct_k + dg_k$ is good, for every $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in S, k \geq 1$ if and only if $\text{Cone}(S)$ satisfying P1) possesses the same property. The first assertion is obvious and the second one follows from the fact that the sign of the leading term of $F_k^\sigma M) = af_k^\sigma + bh_k^\sigma + ct_k^\sigma + dg_k^\sigma$ with $a > 0$ is independent of a (see the proof of Lemma 2.5)

We shall prove several lemmas before proving this proposition.

2.2. Definition of M^{ij} . Let σ be a sequence of $2k$ signs and let $\sigma_i, i = 0, 1, 2, 3$, be the sequence of $2k+2$ signs such that (a) $\sigma_i(j) = \sigma(j)$, for each $1 \leq j \leq 2k$ and (b) $\sigma_0(2k+1) = 1, \sigma_0(2k+2) = 1, \sigma_1(2k+1) = 1, \sigma_1(2k+2) = -1, \sigma_2(2k+1) = -1, \sigma_2(2k+2) = 1, \sigma_3(2k+1) = -1, \sigma_3(2k+2) = -1$. Obviously every sequence of $2k+2$ signs equals to σ_i , for some σ and

i. For any $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, set

$$\begin{aligned} M^{00} &= 4A_1M, & M^{01} &= 2A_3M, & M^{02} &= 2A_2^tM, & M^{03} &= A_4^tM \\ M^{10} &= 4A_1M, & M^{11} &= 2A_2M, & M^{12} &= 2A_2^tM, & M^{13} &= A_5M \\ M^{20} &= 4A_1M, & M^{21} &= 2A_3M, & M^{22} &= 2A_3^tM, & M^{23} &= A_6M \\ M^{30} &= 4A_1M, & M^{31} &= 2A_2M, & M^{32} &= 2A_3^tM, & M^{33} &= A_4M. \end{aligned}$$

Lemma 2.4. For any $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \text{Mat}(2, 2)$ and $k \geq 1$,

(2.3)

$$\text{tr}(F_{k+1}^{\sigma_i} M) = (-1)^{\tau(i)} (x_{k+1}y_{k+1}\text{tr}(F_k^\sigma M^{i0}) + x_{k+1}\text{tr}(F_k^\sigma M^{i1}) + y_{k+1}\text{tr}(F_k^\sigma M^{i2}) + \text{tr}(F_k^\sigma M^{i3}))$$

where $i = 0, 1, 2, 3$, $\tau(0) = \tau(3) = 1$, $\tau(1) = \tau(2) = 0$.

Proof. For $i = 0$,

$$\begin{aligned} F_{k+1}^{\sigma_0} &= F_k^\sigma A^{1+x_{k+1}} B^{1+y_{k+1}} = F_k^\sigma \begin{pmatrix} 1 & 2+2x_{k+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2-2y_{k+1} & 1 \end{pmatrix} \\ &= F_k^\sigma \begin{pmatrix} -3-4x_{k+1}-4y_{k+1}-4x_{k+1}y_{k+1} & 2+2x_{k+1} \\ -2-2y_{k+1} & 1 \end{pmatrix} \\ &= -(x_{k+1}y_{k+1}F_k^\sigma(4A_1) + x_{k+1}F_k^\sigma(2A_3) + y_{k+1}F_k^\sigma(2A_2^t) + F_k^\sigma A_4^t) \end{aligned}$$

So, (2.3) holds for $i = 0$. Similarly for $i = 1, 2, 3$. □

Lemma 2.5. Let σ be a sequence of $2k$ signs and $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, with $a > 0$ and assume further that the $(1, 1)$ entry of M^{ij} is also positive for every i, j . Then for any $0 \leq i \leq 3$,

all the coefficients of $\text{tr}(F_{k+1}^{\sigma_i} M)$ are of the same sign if and only if all the coefficients of $\text{tr}(F_k^{\sigma} M^{ij})$ are of the same sign, for $j = 0, 1, 2, 3$.

Proof. Note that the explicit formula of f_k, h_k, t_k, g_k in [1] implies that $\deg(h_k) = \deg(t_k) = 2k - 1, \deg(g_k) = 2k - 2, \deg(f_k) = 2k$ and the leading term of f_k is $(-1)^k 4^k x_1 y_1 \cdots x_k y_k$. Hence if all the coefficients of $\text{tr}(F_k^{\sigma} M) = af_k^{\sigma} + bh_k^{\sigma} + ct_k^{\sigma} + dg_k^{\sigma}$, with $a > 0$, are of the same sign, then all the coefficients have the same sign with $(-1)^{k+\sharp(\sigma)}$ where $\sharp(\sigma)$ is the number of negative signs that σ takes. Now the lemma follows immediately from (2.3). \square

We have $f_1 = 1 - 4x_1 y_1, h_1 = 2x_1, t_1 = -2y_1, g_1 = 1$. If set $F_0 = E$, the identity matrix, then (2.3) also holds for $k = 0$, i.e.

$$(2.4) \quad \text{tr}(F_1^{\sigma_i} M) = (-1)^{\tau(i)} (x_1 y_1 \text{tr}(M^{i0}) + x_1 \text{tr}(M^{i1}) + y_1 \text{tr}(M^{i2}) + \text{tr}(M^{i3}))$$

where the sequences $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ of 2 signs are respectively $\{+, +\}, \{+, -\}, \{-, +\}, \{-, -\}$.

Let $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Lemma 2.6. *For $a > 0$ and assume the $(1, 1)$ entry of M^{ij} is positive, then $af_1 + bh_1 + ct_1 + dg_1$ is good if and only if $\text{tr}(M^{i3}) \geq 0, i = 0, 1, 2, 3$.*

Proof. It is easy to see by (2.4) that $af_1 + bh_1 + ct_1 + dg_1$ is good if and only if $\text{tr}(M^{ij}) \geq 0$, for all i, j . Now the lemma follows immediately from (2.1). \square

2.3. Proof of Proposition 2.2. We prove it by induction on k . For $k = 1$, $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in S$, we have $af_1 + bh_1 + ct_1 + dg_1$ is good by Lemma 2.6. Now assume $af_k + bh_k + ct_k + dg_k$ is good, for all $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in S$. One deduces first that all M^{ij} are contained $\text{Cone}(S)$ by (2.1) and the condition p3) that S satisfies, and then that $\text{tr}(F_k M^{ij})$ is good by the induction hypothesis and Remark 2.3. Therefore $af_{k+1} + bh_{k+1} + ct_{k+1} + dg_{k+1}$ is good as well, by Lemma 2.5. \square

3. PROOF OF THE MAIN THEOREM

3.1. Decreasing matrices. A matrix $X = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is said to be decreasing if $|a| > |b| > |d|$ and $|a| > |c| > |d|$, according to [1]. The following lemma is proved in [1].

Lemma 3.1. *Let $X \in \Gamma(2)$ be decreasing and $m, n \in \mathbf{Z} \setminus \{0\}$, then $Y = A^m B^n X$ is decreasing.*

3.2. Proof of Theorem 1.1. Let $\Delta = \{K_1 K_2 \cdots K_n \mid n \geq 0, K_i = A_4, A_4^t, A_5, A_6\}$ (for $n = 0$, we mean the identity matrix E). By (2.2) every matrix except the identity matrix in Δ is decreasing by Lemma 3.1. Now assume every word of length n , $M = K_1 K_2 \cdots K_n = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ in Δ , has the property that $a > 0$. Then for any word in Δ of length $n + 1$, $M' = K_1 K_2 \cdots K_{n+1} = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$, it is easy to show that $a' > 0$ since M is decreasing and $M' = M K_{n+1}$ with $K_{n+1} \in \{A_4, A_4^t, A_5, A_6\}$. Hence we have proved, by induction, that for every $C = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \Delta$, $a > 0$, that is, Δ satisfies the condition P1).

In addition it is easy to see that the trace of a decreasing matrix whose $(1, 1)$ entry is positive is always positive. Thus Δ satisfies the condition P2) as well. Meanwhile Δ obviously satisfies the conditions P3) by the definition of Δ . Therefore $af_k + bh_k + ct_k + dg_k$ is good, for all $k, \forall M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \Delta$, by Proposition 2.2. Now Theorem 1.1 follows. \square

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